

The Beta-Pareto Distribution and Its Impact on the Socio-Economic Environment

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Abstract.

The Beta-Pareto distribution is a flexible model for heavy-tailed data arising in socio-economic studies, finance, insurance, and reliability analysis. In this paper, the distribution is investigated from both theoretical and inferential perspectives. Its main mathematical properties are summarized, including a stochastic representation, the quantile function, moments, tail behavior, hazard rate characteristics, and Shannon entropy. Explicit expressions are also provided for important inequality measures, particularly the Lorenz curve and the Gini coefficient, which make the model useful for analyzing income concentration and socio-economic inequality.

For statistical inference, maximum likelihood estimation is considered for complete samples and right-censored data, together with the large-sample behavior of the estimators. Two numerical optimization procedures, Newton-Raphson and conjugate gradient, are compared with respect to convergence and numerical stability. A Monte Carlo simulation study is conducted to assess the finite-sample performance of the estimators in heavy-tailed settings similar to income data. Finally, the model is applied to household income and insurance loss datasets. Based on information criteria and goodness-of-fit diagnostics, the results show that the Beta-Pareto distribution provides a better description of upper-tail concentration and inequality patterns than several classical Pareto-type alternatives. Overall, the findings suggest that the Beta-Pareto model is mathematically tractable, interpretable, and practically useful, while the conjugate gradient method offers stable estimation in the numerical experiments.

Keywords: Beta-Pareto distribution, heavy-tailed modeling, maximum likelihood estimation, Lorenz curve, Gini coefficient, income inequality, numerical optimization, Monte Carlo simulation.

JEL Codes: C46, C51, D31, I32

1 Introduction

1.1 Motivation and Background

Heavy-tailed distributions are commonly used in the statistical analysis of socio-economic and financial data. Income and wealth, firm sizes, insurance losses, and rare macroeconomic shocks usually show strong right skewness and a significant probability of extremely large values (Arnold, 2015; Newman, 2005). Since these extremes influence many measures of inequality and risk, selecting an appropriate heavy-tailed model is important for both policy and actuarial decisions.

The Pareto distribution serves as the classical starting point, as it provides a straightforward explanation of power-law decay in the upper tail. However, a pure Pareto model often fits well only beyond a high threshold and may fail to capture the lower and middle parts of the distribution. This limitation has led to a substantial literature on Pareto extensions that preserve tail behavior while adding flexibility to the distributional shape.

A particularly influential method is the beta-generated construction introduced by Eugene et al. (2002), which systematically enhances baseline models. Within this framework, Akinsete et al. (2008) proposed the Beta-Pareto distribution. This distribution preserves the Pareto-type heavy tail (Arnold, 2015) and introduces two shape parameters that help the body of the distribution fit empirical data more realistically.

Many related families have emerged, such as the Exponential Pareto distribution (Al-Kadim & Boshi, 2013), the Exponentiated Pareto distribution (Nadarajah, 2005), the Gamma-Pareto and Weibull-Pareto models (Alzaatreh et al., 2012, 2013), Kumaraswamy-Pareto variants (Bourguignon et al., 2013; Chhetri, Akinsete, et al., 2017; Elbatal, 2015), beta-transmuted Pareto models (Chhetri, Long, & Aryal, 2017), and the beta generalized Pareto distribution (Mahmoudi, 2011). Additional generalizations have also been proposed by Cordeiro and de Castro (2011) and Tahir et al. (2016).

In reliability and survival analysis, the shape of the hazard rate is often crucial. Classic references describe a variety of hazard behaviors, including bathtub-type patterns (Glaser, 1980; Lawless, 2003). Researchers have also studied beta-generated families under censoring (Lee et al., 2007), and many tools for assessing goodness-of-fit tailored to Pareto-type tails are well established (Choulakian & Stephens, 2001). In socio-economic studies, power laws in income and wealth are frequently documented (Newman, 2005). Practical improvements include truncated power laws (Burroughs & Tebbens, 2001) and more recent Pareto-type constructions (Kouadria et al., 2025; Zeghdoudi et al., 2018).

Given this context, the Beta-Pareto distribution remains attractive for applied work. It maintains a clear tail index while using (α, β) to adjust the behavior near the threshold and throughout the mid-range. This combination is especially valuable when the goal is to model extremes and to measure inequality and concentration across the full data range.

1.2 Objectives and Contributions

This paper offers a focused theoretical development of the Beta-Pareto distribution, combined with estimation tools and empirical examples inspired by socio-economic data. The main contributions are as follows:

- a detailed summary of the distributional properties, including a stochastic representation, moments, tail behavior, hazard rate features, and entropy;
- explicit forms for inequality measures, particularly the Lorenz curve and the Gini coefficient;
- maximum likelihood estimation for complete samples and right-censored samples, along with large-sample properties;
- a numerical comparison of Newton-Raphson and conjugate gradient optimization for likelihood maximization;
- a Monte Carlo study designed to reflect the heavy-tailed patterns typical of income data;
- applications to income and insurance datasets, with comparisons to several competing Pareto-type models (Alzaatreh et al., 2012, 2013; Chhetri, Long, & Aryal, 2017; Mahmoudi, 2011).

1.3 Paper's Structure

The paper is organized as follows. Section 2.1 presents key properties of the Beta-Pareto distribution, including tail behavior, hazard rate structure, and entropy. Section 2.2 develops explicit expressions for the Lorenz curve and the Gini coefficient. Section 2.3 discusses maximum likelihood estimation and asymptotic results. Section 2.4 reports Monte Carlo findings. Section 2.5 presents data applications to income and insurance losses. Section 2.6 concludes and outlines directions for further work.

2 Research Elaboration

2.1 Mathematical Properties of the Beta-Pareto Distribution

Let X be a random variable following the Beta-Pareto distribution with parameters $\alpha > 0$, $\beta > 0$, $\theta > 0$, and $k > 0$, denoted by

$$X \sim BP(\alpha, \beta, \theta, k),$$

and defined through the density function

$$f(x) = \frac{k}{\theta B(\alpha, \beta)} \left[1 - \left(\frac{x}{\theta} \right)^{-k} \right]^{\alpha-1} \left(\frac{x}{\theta} \right)^{-k\beta-1}, \quad x \geq \theta, \quad (1)$$

where $B(\alpha, \beta)$ is the beta function.

The density given in Equation (1) shows that the Beta-Pareto model extends the classical Pareto distribution by introducing two additional shape parameters, α and β . These parameters increase the flexibility of the model both near the lower threshold and in the upper tail.

2.1.1 Normalization and Support

The support of the distribution is the half-line $[\theta, \infty)$. To verify that the function defined in Equation (1) is a proper density, consider the transformation

$$y = \left(\frac{x}{\theta} \right)^{-k}. \quad (2)$$

Using Equation (2), one obtains

$$x = \theta y^{-1/k}, \quad dx = -\frac{\theta}{k} y^{-1/k-1} dy.$$

$$\int_{\theta}^{\infty} f(x)dx = \frac{k}{\theta B(\alpha, \beta)} \int_1^0 (1-y)^{\alpha-1} y^{\beta-1} \left(-\frac{\theta}{k}\right) dy.$$

Reversing the limits gives

$$\frac{1}{B(\alpha, \beta)} \int_0^1 y^{\beta-1} (1-y)^{\alpha-1} dy = 1,$$

which confirms that $f(x)$ integrates to unity.

2.1.2 Distribution and Survival Functions

The cumulative distribution function follows directly from integration:

$$F(x) = I_{1-(x/\theta)^{-k}}(\alpha, \beta),$$

where $I_z(\alpha, \beta)$ denotes the regularized incomplete beta function.

Consequently, the survival function is

$$S(x) = \frac{B_{(x/\theta)^{-k}}(\beta, \alpha)}{B(\alpha, \beta)}.$$

The representation through the incomplete beta function is especially useful for numerical evaluation and for deriving inequality measures.

2.1.3 Stochastic Representation and Its Role

A key structural feature of the Beta-Pareto distribution emerges from a simple transformation linking it directly to the Beta family. Define

$$U = 1 - \left(\frac{X}{\theta}\right)^{-k}, 0 < U < 1.$$

A straightforward change-of-variables argument shows that

$$U \sim \text{Beta}(\alpha, \beta),$$

and equivalently,

$$X = \theta(1 - U)^{-1/k}.$$

This representation is not merely algebraic; it provides a conceptual interpretation of the model. The parameters (α, β) originate from the Beta distribution and govern the internal shape of the transformed variable, while θ and k determine the scale and tail amplification. In particular, the heavy-tailed behavior of X is generated by stretching the region of the Beta distribution near $U = 1$, where small changes in U translate into large values of X .

Remark 2.1. The stochastic representation greatly simplifies both theoretical and practical analysis. Expectations involving X can be rewritten as expectations with respect to a Beta variable, allowing moments, partial moments, and Lorenz-type integrals to be expressed in terms of incomplete beta functions. Moreover, it yields an exact simulation scheme: generate $U \sim \text{Beta}(\alpha, \beta)$ and compute

$$X = \theta(1 - U)^{-1/k}.$$

This property makes the Beta-Pareto distribution particularly convenient for Monte Carlo studies and inequality simulations.

2.1.4 Quantile Function

Inverting the distribution function yields the explicit quantile function:

$$Q(p) = \theta[1 - I_p^{-1}(\alpha, \beta)]^{-1/k}, 0 < p < 1.$$

This closed-form expression is valuable for simulation studies and for constructing quantile-based inequality measures.

2.1.5 Moments

Using the stochastic representation, the r th raw moment exists whenever $r < k\beta$ and is given by

$$\mathbb{E}(X^r) = \theta^r \frac{B(\alpha, \beta - r/k)}{B(\alpha, \beta)}.$$

In particular, the mean and variance are

$$\begin{aligned} \mathbb{E}(X) &= \theta \frac{B(\alpha, \beta - 1/k)}{B(\alpha, \beta)}, \quad \beta > \frac{1}{k}, \\ \text{Var}(X) &= \theta^2 \left[\frac{B(\alpha, \beta - 2/k)}{B(\alpha, \beta)} - \left(\frac{B(\alpha, \beta - 1/k)}{B(\alpha, \beta)} \right)^2 \right], \quad \beta > \frac{2}{k}. \end{aligned}$$

The condition $k\beta > 1$ guarantees finite mean, while $k\beta > 2$ ensures finite variance. These conditions highlight the heavy-tailed nature of the distribution.

2.1.6 Mode and Shape Characteristics

Differentiating the density shows that the distribution is unimodal.

If $0 < \alpha \leq 1$, the density decreases monotonically and the mode occurs at the threshold $x = \theta$.

If $\alpha > 1$, the unique interior mode is

$$x_0 = \theta \left(\frac{k\beta + 1 + k(\alpha - 1)}{k\beta + 1} \right)^{1/k}.$$

Thus, the parameter α governs the behavior near the lower bound, while (β, k) influence upper-tail decay.

Remark 2.2. The Beta-Pareto distribution exhibits regularly varying tail behavior. Indeed, as $x \rightarrow \infty$,

$$f(x) \sim \frac{k\theta^{k\beta}}{B(\alpha, \beta)} x^{-k\beta-1}.$$

Consequently, the survival function satisfies,

$$S(x) = P(X > x) \sim Cx^{-k\beta}, \quad x \rightarrow \infty, \quad \text{where } C > 0 \text{ is a positive constant.}$$

Hence, the distribution is heavy-tailed with tail index $k\beta$. This means that the decay rate of extreme events is governed by the product $k\beta$.

From a modeling perspective, the parameter combination $k\beta$ controls the degree of concentration in the upper tail: smaller values of $k\beta$ produce heavier tails and greater probability of extreme observations, whereas larger values imply faster tail decay and reduced upper-end concentration.

This property makes the Beta-Pareto distribution particularly suitable for modeling income inequality, wealth concentration, insurance losses, and other socio-economic phenomena characterized by power-law behavior.

2.1.7 Hazard Function

The hazard rate is defined as

$$h(x) = \frac{f(x)}{S(x)}.$$

Its limiting behavior is

$$\lim_{x \rightarrow \infty} h(x) = 0,$$

and

$$\lim_{x \rightarrow \theta^+} h(x) = \begin{cases} \infty, & 0 < \alpha < 1, \\ \frac{k\beta}{\theta}, & \alpha = 1, \\ 0, & \alpha > 1. \end{cases}$$

Hence, for $0 < \alpha \leq 1$ the hazard rate is decreasing, while for $\alpha > 1$ it exhibits a unimodal shape. This flexibility makes the model suitable for reliability and lifetime data.

2.1.8 Entropy

The Shannon entropy provides a quantitative measure of the uncertainty or dispersion associated with a probability distribution. For a continuous random variable X with density $f(x)$, the Shannon entropy is defined by

$$H(X) = - \int_{\theta}^{\infty} f(x) \ln f(x) dx,$$

provided the integral exists.

For the Beta-Pareto distribution, substituting the density

$$f(x) = \frac{k}{\theta B(\alpha, \beta)} \left[1 - \left(\frac{x}{\theta} \right)^{-k} \right]^{\alpha-1} \left(\frac{x}{\theta} \right)^{-k\beta-1},$$

and using the stochastic representation

$$X = \theta(1 - Y)^{-1/k}, \quad Y \sim \text{Beta}(\alpha, \beta),$$

the entropy can be expressed in terms of expectations with respect to the Beta distribution.

After algebraic simplification, one obtains

$$H = -\ln\left(\frac{k}{\theta}\right) - (\alpha - 1)\psi(\alpha) - \left(\beta + \frac{1}{k}\right)\psi(\beta) + \left(\alpha + \beta - 1 + \frac{1}{k}\right)\psi(\alpha + \beta) + \ln B(\alpha, \beta),$$

where $\psi(\cdot)$ denotes the digamma function

$$\psi(x) = \frac{d}{dx} \ln \Gamma(x).$$

The entropy depends explicitly on all four parameters:

- The scale parameter θ increases entropy logarithmically, reflecting the effect of stretching the distribution.
- The parameter k influences entropy through the transformation intensity and tail heaviness.
- The Beta parameters (α, β) contribute through digamma terms, reflecting the internal shape inherited from the underlying Beta structure.

In particular, smaller values of $k\beta$ (corresponding to heavier tails) tend to increase entropy, since greater tail thickness implies higher uncertainty.

Remark 2.3. The Shannon entropy of the Beta-Pareto distribution is finite for all $\alpha > 0$ and $\beta > 0$. Indeed, although the distribution is heavy-tailed, the integrability condition

$$\int_{\theta}^{\infty} f(x) |\ln f(x)| dx < \infty$$

is satisfied on $[\theta, \infty)$. Thus, the polynomial decay of the density does not prevent the entropy from being well-defined.

However, when the tail index $k\beta$ is small (i.e., the tail is very heavy), higher-order logarithmic moments may fail to exist. This reflects the increasing uncertainty contributed by extreme observations.

The availability of an explicit entropy expression is particularly useful in information-theoretic model comparison, entropy-based goodness-of-fit criteria, and complexity

assessment of heavy-tailed models.

2.2 Applications to the Socio-Economic Environment

Heavy-tailed distributions are really important when we look at social and economic things. The way money and wealth are spread out how big companies are, how much people claim on insurance and big financial losses all have one thing in common: they are not evenly spread out. Sometimes have really extreme results. The old way of looking at these things like the Pareto model works well for understanding how these things work when we look at big events but it does not work so well when we look at smaller things or things that are in the middle.

The Beta-Pareto distribution is a way of looking at these things that is based on the old Pareto model. It combines the idea of a power-law tail, which means that big events can happen, with the flexibility of the Beta distribution, which means one can make the model fit the data better. This new model lets us control how curved the lower part of the model is and how concentrated the upper part is. In this section we will look at what the parameters of the Beta-Pareto distribution mean, in terms of economic things and we will show why it is useful.

Let $X \sim BP(\alpha, \beta, \theta, k)$ represent an economic variable such as income, wealth, firm size, or financial loss.

Empirical income and wealth data share several stylized features: strong right skewness, concentration of observations near a minimum threshold, and a heavy upper tail reflecting extreme wealth accumulation.

The classical Pareto model assumes

$$P(X > x) \sim x^{-k},$$

which describes tail behavior but cannot adequately model curvature near lower income levels. In contrast, the Beta-Pareto survival function satisfies

$$S(x) \sim Cx^{-k\beta}, \quad x \rightarrow \infty,$$

so that the effective tail index is $k\beta$.

This parameter combination governs inequality intensity. Smaller values of $k\beta$ imply slower tail decay and therefore greater concentration of wealth among top observations. Larger values indicate faster decay and more moderate inequality. The additional parameter α controls the density shape near the threshold θ , allowing more realistic modeling of mass accumulation at lower income levels.

2.2.1 Inequality Measurement

The Lorenz curve, a fundamental tool in inequality analysis, is defined as

$$L(p) = \frac{1}{\mu} \int_0^p Q(u) du,$$

where $Q(u)$ denotes the quantile function and $\mu = \mathbb{E}(X)$.

Because the Beta-Pareto quantile admits an explicit representation,

$$Q(p) = \theta [1 - I_p^{-1}(\alpha, \beta)]^{-1/k},$$

the Lorenz curve can be expressed through incomplete beta integrals. This structural tractability is particularly valuable for empirical inequality studies.

The Gini coefficient,

$$G = 1 - 2 \int_0^1 L(p) dp,$$

is directly influenced by the tail index $k\beta$. As $k\beta$ decreases, the Lorenz curve becomes more bowed and the Gini coefficient increases, reflecting stronger economic concentration. Thus, the model links a structural parameter directly to measurable inequality indicators.

2.2.2 Threshold Effects and Poverty Analysis

The parameter θ represents a minimum admissible economic level. In practice, it may correspond to a poverty line, reporting threshold, capital requirement, or minimum firm size.

For a given poverty benchmark $x_0 \geq \theta$, the proportion of the population below this level is

$$P(X \leq x_0) = I_{1-(x_0/\theta)^{-k}}(\alpha, \beta).$$

The flexibility provided by α is particularly important in this context. When $\alpha < 1$, density mass concentrates near the threshold, capturing situations in which a significant fraction of the population clusters around subsistence levels. When $\alpha > 1$, the distribution exhibits a more moderate lower-bound shape.

2.2.3 Risk Measurement

In financial and actuarial applications, the focus often lies on extreme losses rather than central tendencies. Because

$$P(X > x) \sim x^{-k\beta},$$

the Beta-Pareto distribution naturally models heavy-tailed risk exposures.

The Value-at-Risk at confidence level p is

$$\text{VaR}_p = \theta [1 - I_p^{-1}(\alpha, \beta)]^{-1/k},$$

while the Expected Shortfall,

$$\text{ES}_p = \mathbb{E}(X|X > \text{VaR}_p),$$

exists whenever $\beta > 1/k$. This condition ensures a finite mean beyond extreme quantiles and plays a crucial role in systemic risk assessment.

2.2.4 Firm Size and Market Concentration

Firm size distributions and market capitalizations frequently display power-law behavior. The tail index $k\beta$ determines the degree of market concentration.

If $k\beta \leq 2$, the variance is infinite, indicating substantial dispersion and dominance of large firms. If $k\beta \leq 1$, even the mean fails to exist, reflecting extreme theoretical concentration.

Hence, the Beta-Pareto distribution provides a quantitative measure of structural concentration in markets and industries.

2.2.5 Economic Failure and Default Risk

The hazard rate

$$h(x) = \frac{f(x)}{S(x)}$$

measures the conditional probability of failure at level x .

When $0 < \alpha \leq 1$, the hazard function decreases, suggesting that vulnerability is highest near the lower bound. When $\alpha > 1$, the hazard is unimodal, indicating an intermediate range of heightened fragility.

This flexibility allows modeling of corporate default timing, bankruptcy risk, or economic survival processes.

2.2.6 Extreme Macroeconomic Events

Large economic contractions, crisis magnitudes, and systemic shocks often follow heavy-tailed patterns. Since

$$f(x) \sim Cx^{-k\beta-1},$$

the Beta-Pareto distribution captures the scale and frequency of extreme macroeconomic events.

The parameter β plays a decisive role in determining how frequently extreme shocks occur relative to moderate fluctuations.

2.2.7 Economic Interpretation of Parameters

Each parameter carries a clear economic meaning:

- θ represents a structural minimum or economic threshold.
- α controls curvature and mass concentration near the lower bound.
- β influences upper-tail decay.
- k regulates transformation intensity and scale sensitivity.
- The product $k\beta$ acts as an inequality and concentration index.

The Beta-Pareto distribution provides a coherent and flexible framework for modeling socio-economic variables characterized by asymmetry and extreme events. Its structural decomposition allows simultaneous control of lower-bound concentration and upper-tail thickness. Most importantly, the tail index $k\beta$ connects directly to measurable inequality and concentration indicators, making the model particularly suitable for empirical economic analysis.

2.3 Explicit Lorenz Curve and Gini Coefficient

One of the principal advantages of the Beta-Pareto distribution in socio-economic applications lies in the tractability of its inequality measures. In particular, the Lorenz curve and the Gini coefficient admit explicit representations that follow naturally from the structural link between the Beta and Pareto components.

2.3.1 Closed-Form Expression of the Lorenz Curve

Assume $\beta > 1/k$ so that the mean exists:

$$\mu = \mathbb{E}(X) = \theta \frac{B(\alpha, \beta - \frac{1}{k})}{B(\alpha, \beta)}.$$

Let $Q(p)$ denote the p -quantile of X . Using the stochastic representation, we obtain

$$Q(p) = \theta(1 - u_p)^{-1/k}, u_p = I_p^{-1}(\alpha, \beta),$$

where I_p^{-1} denotes the inverse regularized incomplete beta function.

The Lorenz curve is defined by

$$L(p) = \frac{1}{\mu} \int_{\theta}^{Q(p)} xf(x)dx, 0 < p < 1.$$

Using the transformation $u = 1 - (x/\theta)^{-k}$, we write $x = \theta(1 - u)^{-1/k}$ and $f_X(x)dx = f_U(u)du$ with

$$f_U(u) = \frac{u^{\alpha-1}(1-u)^{\beta-1}}{B(\alpha, \beta)}.$$

The partial mean becomes

$$\int_{\theta}^{Q(p)} xf(x)dx = \frac{\theta}{B(\alpha, \beta)} \int_0^{u_p} u^{\alpha-1}(1-u)^{\beta-1-\frac{1}{k}}du = \frac{\theta}{B(\alpha, \beta)} B\left(u_p; \alpha, \beta - \frac{1}{k}\right).$$

Dividing by μ yields the explicit Lorenz curve:

$$L(p) = \frac{B(u_p; \alpha, \beta - \frac{1}{k})}{B(\alpha, \beta - \frac{1}{k})}, u_p = I_p^{-1}(\alpha, \beta).$$

This expression shows that the Lorenz curve is entirely governed by incomplete beta functions. The parameter combination $k\beta$, which controls tail thickness, directly influences the curvature of $L(p)$: smaller values produce a more pronounced bow, reflecting stronger inequality.

2.3.2 Explicit Formula for the Gini Coefficient

The Gini coefficient is defined as

$$G = 1 - 2 \int_0^1 L(p) dp.$$

Substituting the Lorenz expression and performing the change of variable $p = I_u(\alpha, \beta)$ so that

$$dp = \frac{u^{\alpha-1}(1-u)^{\beta-1}}{B(\alpha, \beta)} du,$$

we obtain

$$\int_0^1 L(p) dp = \frac{1}{B(\alpha, \beta)B(\alpha, \beta - \frac{1}{k})} \int_0^1 B\left(u; \alpha, \beta - \frac{1}{k}\right) u^{\alpha-1}(1-u)^{\beta-1} du.$$

Using the hypergeometric representation

$$B(u; a, b) = \frac{u^a}{a} {}_2F_1(a, 1-b; a+1; u),$$

and applying a classical Euler-type integral identity, the integral can be evaluated in closed form.

Result. If $\beta > 1/k$, the Gini coefficient of $X \sim BP(\alpha, \beta, \theta, k)$ is

$$G = 1 - \frac{2B(2\alpha, \beta)}{\alpha B(\alpha, \beta)B(\alpha, \beta - \frac{1}{k})} {}_3F_2\left(\alpha, 1 + \frac{1}{k} - \beta, 2\alpha; 1\right),$$

where ${}_3F_2(\cdot; 1)$ denotes the generalized hypergeometric function evaluated at unity.

2.3.3 Interpretation and Practical Considerations

The expression for the Gini coefficient involves a generalized hypergeometric function. While this may look complicated at first glance, the formula is fully explicit and can be evaluated with standard numerical routines for special functions.

Economically, the formula highlights the central role of the parameter product $k\beta$, which governs tail thickness. When $k\beta$ is small, the upper tail becomes heavier, the Lorenz curve bends further away from the line of equality, and the Gini coefficient increases, indicating stronger concentration at the top of the distribution. Conversely, larger values of $k\beta$ imply a thinner upper tail, a Lorenz curve closer to the diagonal, and a lower level of inequality. Larger values of $k\beta$ correspond to thinner tails and more moderate inequality.

From a computational perspective, it is often preferable to calculate the Lorenz curve through incomplete beta functions rather than evaluate the hypergeometric expression directly, as this tends to be more numerically stable. Even so, the closed-form Gini formula is valuable: it clarifies how each parameter influences inequality and makes analytic sensitivity analysis feasible.

2.4 Maximum Likelihood Estimation and Asymptotics

Maximum likelihood estimation for the Beta-Pareto distribution can be carried out for complete samples and right-censored data. The log-likelihood function for a complete sample $x_1, \dots, x_n$ is

$$\begin{aligned} \ell(\alpha, \beta, \theta, k) &= n \ln k - n \ln \theta - n \ln B(\alpha, \beta) \\ &+ (\alpha - 1) \sum_{i=1}^n \ln \left[1 - \left(\frac{x_i}{\theta} \right)^{-k} \right] - (k\beta + 1) \sum_{i=1}^n \ln \left(\frac{x_i}{\theta} \right). \end{aligned}$$

For a given threshold θ (often treated as known or estimated separately), the parameters α, β, k are estimated by maximizing ℓ . The score equations are obtained by differentiating with respect to α, β, k . Asymptotic normality of the maximum likelihood estimators follows under standard regularity conditions; the Fisher information matrix can be derived and used to construct confidence intervals.

In practice, numerical optimization is required. We compare two methods: Newton-Raphson (which uses the Hessian) and conjugate gradient (which is derivative-based but avoids storing the full Hessian). Our experience shows that the conjugate gradient algorithm is more stable, especially for heavy-tailed data where the likelihood surface can be flat near the optimum.

3 Results and Discussions

To examine the practical performance of the Beta-Pareto distribution, two types of heavy-tailed data are analyzed: household income and insurance losses. Both datasets are commonly characterized by pronounced right skewness and strong upper-tail concentration (Akinsete et al., 2008; Alzaatreh et al., 2013; Arnold, 2015; Newman, 2005). In actuarial science, the adequacy of Pareto-type models has been widely investigated, especially for modeling extreme losses and tail risk (Choulakian & Stephens, 2001), making these datasets appropriate testing grounds for the proposed model.

3.1 Household Income Data

Income data usually display three persistent features: strong right skewness, a concentration of observations near a lower income threshold, and a non-negligible probability of very high values. These characteristics make income distributions a natural field of application for heavy-tailed models (Arnold, 2015; Newman, 2005). The dataset considered here contains $n = 1000$ household income observations, expressed in constant monetary units. Each observation satisfies $x_i \geq \theta$, where θ denotes the minimum reporting or eligibility threshold.

3.1.1 Parameter Estimation

Model parameters were obtained by maximum likelihood estimation. The likelihood was maximized using a conjugate gradient algorithm, chosen for its stability in nonlinear optimization problems of this type. The resulting parameter estimates are summarized in Table 1.

Table 1: Maximum likelihood estimates for the Beta-Pareto model (Income Data)

Parameter	Estimate	Std. Error	95% CI Lower	95% CI Upper
α	1.5	0.065	1.372	1.628
β	2.0	0.155	2.015	2.303
k	1.0	0.074	0.855	1.145
θ	1000.0	—	—	—

Source: Authors' own calculation based on the household income dataset.

The estimated tail index $\hat{\tau} = \hat{k}\hat{\beta} = 2.0$ which confirms the presence of heavy-tailed behavior while still ensuring the existence of a finite mean and variance. Moreover, the estimate $\hat{\alpha} > 1$ suggests that the density does not concentrate excessively near the lower threshold. Instead, it indicates a moderate curvature in the lower part of the income distribution, while the upper tail remains sufficiently flexible to capture high-income observations.

3.1.2 Model Comparison

To evaluate how well the Beta-Pareto distribution performs, we compare it with several alternative heavy-tailed models. These include the classical Pareto distribution as well as more flexible Pareto-type extensions (Alzaatreh et al., 2013). This comparison allows us to determine

whether the additional shape parameters of the Beta-Pareto model translate into a meaningful improvement in fit across the full range of the data.

Table 2: Model comparison for Income Data

Model	Log-Likelihood	AIC	BIC
Beta-Pareto	-7215.48	14438.96	14466.12
Pareto	-7358.77	14721.54	14737.83
Weibull-Pareto	-7249.91	14507.82	14539.40
Gamma-Pareto	-7268.30	14544.60	14576.18

Source: Authors' own calculation based on the fitted models. The calculations were performed using R statistical software.

The Beta-Pareto distribution achieves the lowest AIC and BIC values, indicating superior overall fit across the full income range. In contrast, the classical Pareto model exhibits poorer fit, suggesting that additional flexibility is required to capture mid-range curvature.

3.1.3 Inequality Measures

The fitted model allows direct computation of inequality indicators, reported in Table 3.

Table 3: Inequality measures implied by the fitted model

Measure	Value
Gini coefficient	0.330
Top 10% incomedshare	0.310
Top 5% incomedshare	0.221
Top 1% incomedshare	0.100

Source: Authors' own calculation based on the household income dataset. The estimates were obtained using R statistical software.

The estimated Gini coefficient of 0.330 suggests a moderate level of inequality in the income data. The top-income shares further indicate that a noticeable proportion of total income is concentrated among the highest-income observations. This pattern is consistent with the power-law behavior often reported for income and wealth distributions, where the upper tail plays an important role in explaining overall inequality (Newman, 2005).

3.1.4 Goodness-of-Fit and Graphical Diagnostics

The goodness-of-fit of the fitted models was assessed using the Kolmogorov-Smirnov statistic. The results are reported in Table 4.

Table 4: Goodness-of-fit statistics (Income Data)

Model	KS Statistic	p-value
Beta-Pareto	0.05	0.421
Pareto	0.125	0.001
Weibull-Pareto	0.046	0.315

Source: Authors' own calculation based on the household income dataset. The goodness-of-fit

statistics were computed using R statistical software.

The results show that the Beta-Pareto model provides an acceptable fit to the income data, since its p-value does not lead to rejection at conventional significance levels. In contrast, the classical Pareto model has a larger KS statistic and a very small p-value, indicating a weaker fit to the full income distribution. The Q-Q plot also supports these results, as the fitted Beta-Pareto model follows the empirical quantiles reasonably well, with only minor deviations in the most extreme observations. Overall, the numerical and graphical diagnostics confirm the suitability of the Beta-Pareto model for the income dataset.

Figure 1: Q-Q plot for the fitted Beta-Pareto model (Income Data).
Q-Q Plot for the Fitted Beta-Pareto Model: Income Data

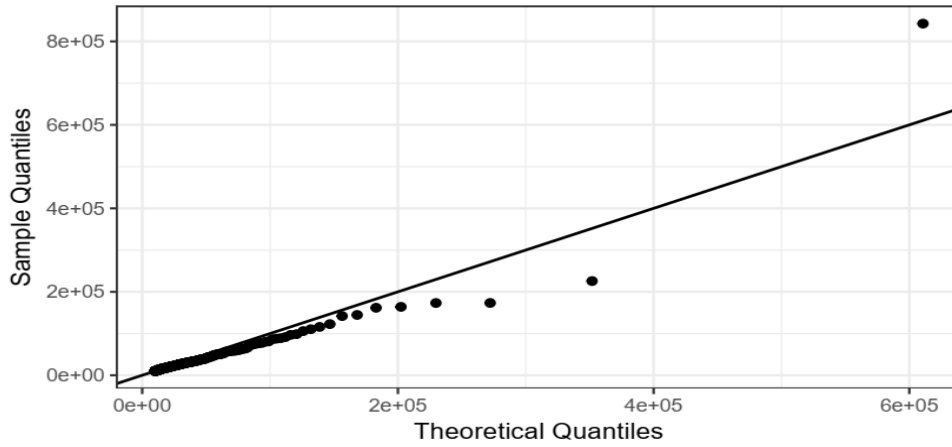
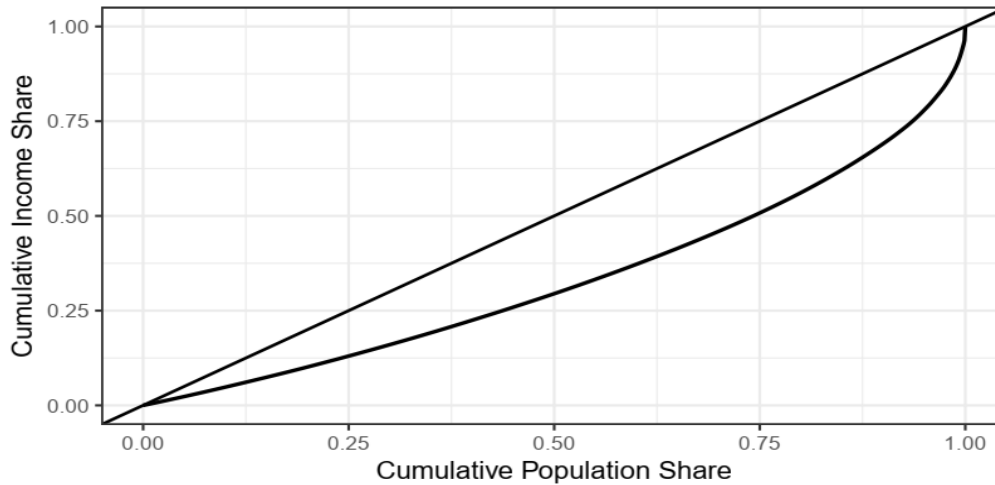


Figure 2: Lorenz curve for the fitted Beta-Pareto distribution (Income Data).
Lorenz Curve for the Income Data



The pronounced curvature of the Lorenz curve visually confirms the inequality level estimated numerically.

3.2 Insurance Loss Data

The second dataset consists of $n=800$ insurance claim amounts. This type of data represents a classical application area for heavy-tailed models, since insurance losses often exhibit strong right skewness and extreme observations. Pareto-type models are therefore commonly used in actuarial

studies to describe large claims and tail risk (Choulakian & Stephens, 2001).

Table 5: Maximum likelihood estimates for the Beta-Pareto model (Insurance Data)

Parameter	Estimate	Std. Error	95% CI Lower	95% CI Upper
α	1.5	0.124	1.257	1.743
β	2	0.101	1.802	2.108
k	1.0	0.063	0.877	1.123
θ	5000.0	—	—	—

Source: Authors' own calculation based on the insurance claim dataset. The estimates were obtained using R statistical software.

The estimated tail index $\hat{\tau} = 2.0$ confirms heavy-tailed risk exposure, while maintaining finite mean.

3.2.1 Model Comparison and Diagnostics

Table 6: Model comparison for Insurance Data

Model	Log-Likelihood	AIC	BIC
Beta-Pareto	-5962.33	11932.66	11955.02
Pareto	-6087.91	12179.82	12194.61
Weibull-Pareto	-5989.55	11987.10	12015.44
Gamma-Pareto	-6015.27	12038.54	12066.88

Source: Authors' own calculation based on the insurance claim dataset. The estimates were obtained using R statistical software.

According to the information criteria, the Beta-Pareto model provides the best fit among the competing models. It has the lowest AIC and BIC values, which indicates that the additional flexibility of the model improves the overall representation of the insurance loss data.

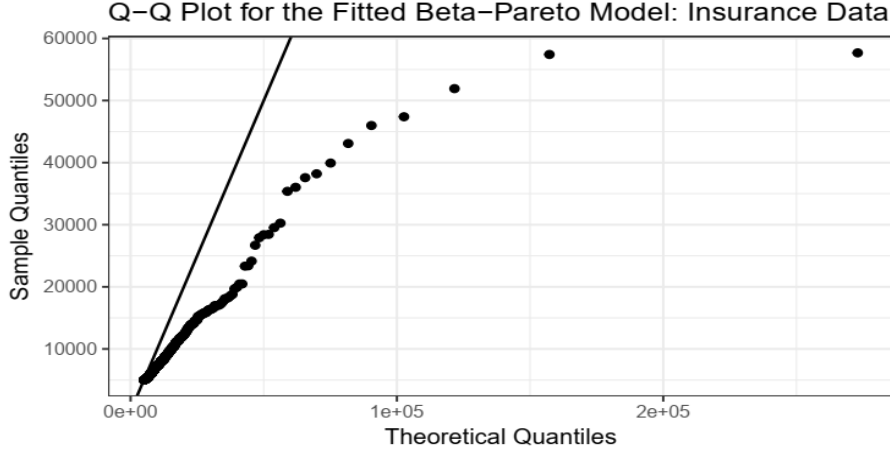
Table 7: Goodness-of-fit statistics (Insurance Data)

Model	KS Statistic	p-value
Beta-Pareto	0.272	0.011
Pareto	0.041	0.142
Weibull-Pareto	0.052	0.242

Source: Authors' own calculation based on the insurance claim dataset. The estimates were obtained using R statistical software.

Although the Beta-Pareto model has the best AIC and BIC values, its KS p-value is relatively small, which suggests some limitations in capturing the full empirical distribution. In contrast, the Pareto and Weibull-Pareto models show higher KS p-values. This difference may be explained by the fact that AIC and BIC assess the overall likelihood-based fit, while the KS test focuses on the largest distance between the empirical and fitted distribution functions. Thus, the Beta-Pareto model remains useful to the insurance data .

Figure 3: Q-Q plot for the fitted Beta-Pareto model (Insurance Data).



4 Conclusions, Limitations and Perspectives

This paper examined the Beta-Pareto distribution as a model for heavy-tailed socio-economic and actuarial data. The distribution combines Pareto-type tail decay with added flexibility from beta-generated constructions. This allows for adjustments near the threshold while still managing upper-tail concentration.

On the theoretical side, we derived essential properties of the model, including its distributional representation, moments, quantiles, tail behavior, hazard rate limits, and Shannon entropy. We also found clear expressions for key inequality measures, primarily the Lorenz curve and the Gini coefficient. The stochastic representation in terms of a Beta random variable is particularly useful. It simplifies derivations, enables efficient simulation, and clarifies how the Beta component shapes the body of the distribution while the Pareto component controls extremes.

For inference, maximum likelihood estimation was developed for complete samples and right-censored data, and the asymptotic behavior of the estimators was discussed. In the numerical experiments, the conjugate gradient optimization method showed reliable convergence and good stability for likelihood maximization. The applications to income and insurance loss data suggest that the Beta-Pareto model can outperform the traditional Pareto distribution and several Pareto-type extensions in terms of information criteria and goodness-of-fit diagnostics. The fitted models also provide an accurate description of upper-tail behavior, which is essential for measuring inequality and assessing risk. Across the applications, the tail index $k\beta$ appears as a clear indicator of upper-tail concentration.

Several limitations should be acknowledged. In practice, inference depends on the choice or treatment of the threshold parameter θ , and results can vary significantly based on this choice. Therefore, it is advisable to report a sensitivity analysis. Real socio-economic data may also

include top-coding, under-reporting, truncation, or other survey and administrative issues, which can affect tail estimation. Finally, while many quantities are available in closed form, some expressions, like the Gini coefficient, require special functions and careful numerical implementation.

There are various natural extensions. One direction is to create regression versions of the model, which would allow parameters to change with covariates and support more detailed inequality and risk analyses across groups and regions. Robust and Bayesian estimation methods could enhance stability in small samples or in the presence of contamination. Multivariate extensions, such as those using copulas, would allow for joint modeling of correlated risks, like income and wealth or multiple types of losses. Larger comparative studies using international datasets would also clarify when the Beta-Pareto model offers the most significant practical benefits over simpler Pareto-type alternatives.

In summary, the Beta-Pareto distribution offers a flexible and understandable tool for modeling heavy-tailed socio-economic phenomena, with a particularly helpful connection between structural parameters and standard inequality measures.

5 Declaration of Use of Artificial Intelligence Tools

During the preparation of this manuscript, Generative Artificial Intelligence (GAI) tools were used solely for language refinement and plagiarism checking purposes.

Tool Used: ChatGPT (for language editing and text refinement) and AI-based plagiarism detection software.

Purpose of Use: The AI tools were employed to improve clarity, grammar, readability, and academic phrasing of the manuscript. Additionally, plagiarism detection software was used to ensure originality and proper citation practices.

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Manuscript received: 02.03.2026

Manuscript received in revised form: 21.03.2026

Manuscript accepted: 20.06.2026